ASYMPTOTIC STABILITY PROPERTIES OF LINEAR VOLTERRA

INTEGRODIFFERENTIAL EQUATIONS

by

R. K. Miller+

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island

+This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGL 40-002-015 and in part by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 693-67B.

I. Introduction

The purpose here is to study the Lyapunov stability properties of solutions of a system of Volterra integrodifferential equations of the form

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds \qquad (' = d/dt) \tag{L}$$

where $t \geq \tau$ and where $x(t) = f(t)$ on $0 \leq t \leq \tau$. Here $\tau \geq 0$ is a given constant, f is a given continuous, vector valued function and both A and $B(t)$ are square matrices. The solution of (L) with initial values (τ, f) will be denoted by $x(t,\tau,f)$. If $\tau = 0$, then the function f reduces to an initial vector $f(0) = x_0$.

Volterra integrodifferential equations occur in a variety of applications. In these applications the initial time τ is always zero. At first sight the initial value problem (L) with $\tau > 0$ appears to be somewhat artificial. Though it may be artificial, it is also useful. For example Grossman and Miller [1] studied the asymptotic behavior of solutions of nonlinear problems of the form

$$y'(t) = A\{y(t)+h_1(y(t))\} + \int_0^t B(t-s)\ \{y(s)+h_2(y(s))\}ds, \quad y(0) = y_0$$

when $|y_0|$ is small. The terms $h_1(y)$ and $h_2(y)$ were assumed to be smooth functions of order $o(|y|)$ as $y \to 0$. The results in [1] depend on certain apriori information about the resolvent $R(t)$ associated with the linear system (L) and its derivative $R'(t)$. In this paper we shall prove that if $B(t) \in L^1(0,\infty)$, then $R(t)$ is of class $L^1(0,\infty)$ if and only if the trivial solution of system (L) is uniformly asymptotically stable. When $B(t)$ and $R(t)$ are both in $L^1(0,\infty)$, then it is easy to see that $R'(t) \in L^1(0,\infty)$ and that $R(t)$ tends to zero as $t \to \infty$. This is exactly the type of information which is necessary in order to apply the results in [1].

The remainder of the paper is organized as follows. Section II contains preliminary definitions and results. In section III we define various types of Lyapunov stability for (L). These definitions are natural extensions of the corresponding notions for ordinary differential equations. Theorem 1 contains general necessary and sufficient conditions for uniform stability and uniform asymptotic stability. The remainder of the section is devoted to connections between stability of (L) and stability properties of a related Volterra integrodifferential equation with infinite memory. In section IV we show that if $B(t)$ and $R(t)$ are in $L^1(0,\infty)$, then the trivial solution of (L) is uniformly asymptotically stable. Moreover for any initial pair (τ,f) the solution $x(t,\tau,f) \in L^1$ on $[\tau,\infty)$. Conversely if $B(t)$ is L^1 and (L) is uniformly asymptotically stable then $R(t)$ must be in $L^1(0,\infty)$. The proof of the converse may be of some independent interest since it depends on constructing a Lyapunov functional for (L).

In sections V and VI we give some sufficient conditions on A and $B(t)$ in order that the trivial solution of (L) is stable, uniformly stable, asymptotically stable or uniformly asymptotically stable. We assume that $B \in L^1(0,\infty)$, that $B^*(s)$ is the Laplace transform of $B(t)$, that is

$$B^*(s) = \int_0^\infty \exp(-st)B(t)dt,$$

and that the determinant of $s - A - B^*(s) \neq 0$ when $\operatorname{Re} s \geq 0$. These assumptions and some additional integrability assumptions on $B(t)$ imply stability of (L). Section VII contains some examples and conjectures.

II. Preliminaries

Let R^n denote real, n-dimensional Euclidean space of column vectors $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$ with the Euclidean norm $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. Let R^+ be the half line $0 \leq t < \infty$. The symbols $C(R^+) = C[0,\infty)$ will denote the set of all continuous functions $\varphi: R^+ \to R^n$ with the topology of uniform convergence on compact subsets of R^+ (the compact-open topology). Given φ in $C(R^+)$ define

$$\|\varphi\|_t = \max\{|\varphi(s)| : 0 \leq s \leq t\}.$$

The symbols $L^P(R^+)$ will denote the usual Lebesgue space of measurable functions f such that

$$\|f\|_P = \{\int_0^\infty |f(t)|^P dt\}^{1/P} < \infty.$$

<u>Definition 1. Let</u> P <u>be a given function with domain</u> $R^+ \times C(R^+)$ <u>and range in</u> R^n. <u>This function</u> $P(t,f)$ <u>will be called nonanticipative if and only if for each</u> $t \geq 0$ <u>one has</u> $P(t,f) = P(t,\varphi)$ <u>wherever</u> f <u>and</u> φ <u>are continuous functions such that</u> $f(s) = \varphi(s)$ <u>on</u> $0 \leq s \leq t$.

Definition 2. Let $P(t,f)$ be a continuous function on $R^+ \times C(R^+)$ into R^n. Then

i) $P(t,f)$ is locally Lipschitz continuous in f if and only if given any pair of positive constants A and B there exists a constant $L > 0$ such that $|P(t,f) - P(t,\varphi)| \leq L\|f-\varphi\|_t$ whenever $0 \leq t \leq A$ and both $|f|_t$ and $|\varphi|_t \leq B$.

ii) $P(t,f)$ is locally Lipschitz continuous in f uniformly in t if it is locally Lipschitz continuous in f and the Lipschitz constants L can be chosen independently of A.

Note that if $P(t,f)$ is continuous in (t,f) and locally Lipschitz continuous in f, then it is automatically nonanticipative.

Consider a system of equations of the form

$$y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds + P(t,y) \tag{2.1}$$

where $t \geq \tau$ and $y(t) = f(t)$ on $0 \leq t \leq \tau$. If $B(t)$ is locally L^1 on R^+ and if $P(t,f)$ is continuous and nonanticipative, then for any pair (τ,f) of initial values in $R^+ \times C(R^+)$ the initial value problem (2.1) has at least one local solution. Any local solution can be continued to the right so long as it remains bounded. If P is locally Lipschitz continuous in f, then the initial values (τ,f) determine the solution of (2.1) uniquely. See Driver [2] for proofs and details of these assertions.

Definition 3. The resolvent $R(t)$ associated with the linear system (L) is the unique solution of the matrix equation

$$\text{(R)} \qquad R'(t) = AR(t) + \int_0^t B(t-s)R(s)ds, \quad R(0) = I$$

<u>where</u> I <u>is</u> <u>the</u> <u>identity</u> <u>matrix</u>.

If $B(t)$ is locally L^1 on R^+, $P(t,f)$ is continuous and non-anticipative and $\tau = 0$, then (2.1) may be rewritten in the equivalent variation of constants form

$$y(t) = R(t)f(0) + \int_0^t R(t-s)P(s,y)ds \quad (t \geq 0).$$

See [1] for details. In particular if $x(t,\tau,f)$ is the solution of (L) for some initial pair (τ,f), then

$$x(t+\tau,\tau,f) = Ax(t+\tau,\tau,f) + \int_0^t B(t-s)x(s+\tau,\tau,f)ds$$
$$+ \int_0^\tau B(t+\tau-s)f(s)ds$$

for $t \geq 0$ with $x(\tau,\tau,f) = f(\tau)$. This inhomogenous initial value problem can be solved with the aid of the variation of constants formula to obtain

$$\text{(2.2)} \qquad x(t+\tau,\tau,f) = R(t)f(\tau) + \int_0^t R(t-s)\left(\int_0^\tau B(s+\tau-u)f(u)du\right)ds$$

for all $t \geq 0$. This form of the solution of (L) will be needed in the sequel.

Let * denote the Laplace transformation. For example

$$\text{(2.3)} \qquad B^*(s) = \int_0^\infty \exp(-st)B(t)dt$$

for all complex numbers s such that the integral exists. The following result was proved in [1].

Lemma 1. Suppose that $B \in L^1(R^+)$ so that (2.3) is defined and continuous when $\operatorname{Re} s \geq 0$. If the determinental condition

$$\text{(D)} \qquad \det(s-A-B^*(s)) \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0$$

is true, then there exists a constant $K > 0$ such that

$$|R^*(s)| \leq K(1+|s|)^{-1} \quad \text{and} \quad |(R')^*(s)| \leq K(1+|s|)^{-1} \quad \text{when} \quad \operatorname{Re} s \geq 0.$$

III. General Stability Considerations

Definition 4. Suppose $B \in L^1(0,T)$ for each $T > 0$. Consider the system (L) with initial conditions $(\tau,f) \in R^+ \times C(R^+)$. The trivial solution $x \equiv 0$ is called:

i) stable if given any $\tau > 0$ and any $\varepsilon > 0$ there exists a number $\delta > 0$ (depending on ε and τ) such that whenever $f \in C(R^+)$ and $\|f\|_\tau \leq \delta$, then the solution $x(t,\tau,f)$ exists for all $t \geq \tau$ and satisfies $|x(t,\tau,f)| \leq \varepsilon$.

ii) uniformly stable if it is stable and δ can be chosen independent of $\tau \geq 0$.

iii) asymptotically stable if it is stable and if given any (τ,f) one has $x(t,\tau,f) \to 0$ as $t \to \infty$.

iv) uniformly asymptotically stable if it is uniformly stable and if given any $\varepsilon > 0$ and any $A > 0$ there exists $T(\varepsilon) > 0$ such that

$|x(t+T(\varepsilon),\tau,f)| \le \varepsilon$ uniformly for all $t \ge \tau$, all $\tau \ge 0$ and all f with $\|f\|_\tau \le A$.

Theorem 1. Suppose $B(t)$ is locally L^1 on R^+. Then:

i) the trivial solution of (L) is uniformly stable if and only if the function $y(t)$ defined by

$$y(t) = \int_0^\infty \left|\int_0^t R(t-s)B(s+u)ds\right| du \tag{3.1}$$

exists and is finite for all $t \ge 0$ and the two functions $R(t)$ and $y(t)$ are uniformly bounded on R^+.

ii) the trivial solution of (L) is uniformly asymptotically stable if and only if it is uniformly stable and both $R(t)$ and $y(t)$ tend to zero as $t \to \infty$.

Proof. Suppose $x \equiv 0$ is uniformly stable. Then there exists a constant B such that for any (τ,f) with $\tau \ge 0$ and $\|f\|_\tau \le 1$ one has $|x(t+\tau,\tau,f)| \le B$ for all $t \ge 0$. If $\tau = 0$, then $|x(t+\tau,\tau,f)| = |x(t,0,f(0))| = |R(t)f(0)| \le B$ for all $t \ge 0$ and all $f(0)$ satisfying $|f(0)| \le 1$. In particular $|R(t)| \le B$ for all $t \ge 0$. Similiarly if $z(t) = x(t+\tau,\tau,f) - R(t)f(\tau)$, then $|z(t)| \le 2B$. By (2.2) $z(t)$ must be of the form

$$\begin{aligned} z(t) &= \int_0^t R(t-s)\left\{\int_0^\tau B(s+\tau-u)f(u)du\right\}ds \\ &= \int_0^t R(t-s)\left\{\int_0^\tau B(s+u)f(\tau-u)du\right\}ds \\ &= \int_0^\tau \left\{\int_0^t R(t-s)B(s+u)ds\right\}f(\tau-u)du. \end{aligned}$$

This means that for all $t,\tau \geq 0$

$$\int_0^\tau \left|\int_0^t R(t-s)B(s+u)ds\right| du \leq 2B.$$

Thus $y(t) \leq 2B$ for all $t \geq 0$.

Conversely if $|R(t)| \leq A$ and $y(t) \leq A$ for some fixed constant A, then by (2.2) one has

$$|x(t+\tau,\tau,f)| \leq |R(t)f(\tau)| + \left|\int_0^\tau \{\int_0^t R(t-s)B(s+u)ds\} f(\tau-u)du\right|$$

$$\leq A|f(\tau)| + A\|f\|_\tau \leq 2A\|f\|_\tau.$$

Thus $x \equiv 0$ is uniformly stable. This proves part i). Part ii) follows in a similiar manner. Q.E.D.

Let $C[-\infty,\infty)$ denote the set of all continuous functions $\varphi\colon R^1 \to R^n$ such that for any $t \in R^1$ the seminorms

(3.2) $$\|\varphi\|_t = \sup\{|\varphi(s)|: -\infty < s \leq t\}$$

are finite. Let $B \in L^1(R^+)$. Consider the initial value problem

(L_∞) $$X'(t) = AX(t) + \int_{-\infty}^t B(t-s)X(s)ds$$

for $t \geq \tau$ with $X(t) = f(t)$ on $-\infty < t \leq \tau$. Here (τ,f) is a pair of initial data in $R^1 \times C[-\infty,\infty)$. The various stability properties for the trivial solution of (L_∞) can be defined in the same way as the corresponding type of stability for (L), see Definition 4. Note that this equation is "autonomous" in the sense that for any (τ,f) one has $X(t,\tau,f) \equiv X(t-\tau,0,f_\tau)$ where f_τ is the translated function $f_\tau(t) = f(t+\tau)$. In particular it follows that one need only consider (L_∞) with initial time $\tau = 0$. Moreover stability and uniform stability are equivalent.

Boundedness or stability for (L_∞) is related to uniform stability for (L). Indeed the following theorem is true.

<u>Theorem 2.</u> <u>Let</u> $B \in L^1(R^+)$. <u>Then all of the following statements are equivalent:</u>

i) <u>the trivial solution of</u> (L) <u>is uniformly stable.</u>

ii) <u>the trivial solution of</u> (L_∞) <u>is (uniformly) stable.</u>

iii) $R(t)$ <u>is bounded and for each</u> $f \in C[-\infty,\infty)$ <u>the solution</u> $X(t,0,f)$ <u>of</u> (L_∞) <u>is bounded on</u> R^+.

<u>Proof.</u> Given initial values $(0,F)$ let $X(t,F) = X(t,0,F)$ be the corresponding solution of (L_∞). Then for any $t \geq 0$ one has

$$X'(t,F) = AX(t,F) + \int_0^t B(t-s)X(s,F)ds + \int_t^\infty B(u)F(t-u)du.$$

If $R(t)$ is the resolvent of (L), then variation of constants yields

$$X(t,F) = R(t)F(0) + \int_0^t R(t-s)\int_s^\infty B(u)F(s-u)duds. \tag{3.3}$$

First suppose that the trivial solution of (L) is uniformly stable. By (3.3) it follows that

$$X(t,F) = R(t)F(0) + \int_0^t R(t-s)\int_0^\infty B(s+u)F(-u)\,du\,ds$$
$$= R(t)F(0) + \int_0^\infty \{\int_0^t R(t-s)B(s+u)ds\}F(-u)du,$$

so that

$$|X(t,F)| \le |R(t)||F(0)| + y(t)\|F\|_0.$$

Here $\|F\|_0$ is defined by (3.2) with $\varphi = F$ and $t = 0$. Since R and y are bounded, this proves that the trivial solution of (L_∞) is stable.

Now assume the stability of (L_∞). Then there exists a constant B such that for any F in $C[-\infty,\infty)$ with $\|F\|_0 \le 1$ one has $|X(t,F)| \le B$ for all $t \ge 0$. Given any unit vector x_0 and any $\varepsilon > 0$ let $F(t) = 0$ if $t \le -\varepsilon$ and $F(t) = (t/\varepsilon + 1)x_0$ if $t \ge -\varepsilon$. Then $\|F\|_0 \le 1$, $|X(t,F)| \le B$ and

$$X(t,F) = R(t)x_0 + \int_0^t R(t-s)\{\int_0^\varepsilon B(s+u)F(-u)du\}ds.$$

In particular one has

$$|R(t)x_0| \le |X(t,F)| + \int_0^t |R(t-s)|\int_0^\varepsilon |B(s+u)|\,du\,ds$$
$$\le B + \int_0^t |R(t-s)|\{\int_s^{s+\varepsilon} |B(u)|du\}ds$$

for all $t > 0$, all $\varepsilon > 0$ and all unit vectors x_0. On letting $\varepsilon \to 0^+$

one finds that $|R(t)x_0| \leq B$ for all t. Thus $|R(t)| \leq B$ uniformly in t. This proves iii).

Finally assume that iii) is true. Given any F, since $X(t,F)$ and $R(t)$ are bounded on R^+ then by (3.3) it follows that

$$A_tF = \int_0^t R(t-s)\int_s^\infty B(u)F(s-u)duds = \int_0^\infty \{\int_0^t R(t-s)B(s+u)ds\}F(-u)du$$

is uniformly bounded in t. For any fixed $t \geq 0$ the symbols A_t represents a bounded linear mapping of $C[-\infty,0]$ into R^n with norm $\|A_t\| = y(t)$. By the principle of uniform boundedness it follows that $\|A_t\| = y(t)$ is uniformly bounded in $t \in R^+$. This provesi). Q.E.D.

The equivalence of i) and iii) in Theorem 2 remains true if in iii) the statement "all $F \in C[-\infty,0]$" is replaced by "all almost periodic F". Similiarly in iii) it would be sufficient to require that $F \in C[-\infty,0]$ and additionally $F(t) \to 0$ as $t \to -\infty$. In general we prove

<u>Theorem 3. Suppose</u> $B \in L^1(R^+)$ <u>and</u> $R(t)$ <u>is bounded. Let</u> Y <u>be a closed, linear subspace of</u> $C[-\infty,0]$ <u>under the uniform norm such that given any</u> f <u>in</u> $C[-\infty,0]$ <u>there exists a sequence</u> f_n <u>in</u> Y <u>such that</u> $\sup\{|f_n(t)| : t \leq 0 \text{ and } n = 1,2,3,\ldots\} < \infty$ <u>and such that</u> $f_n(t) \to f(t)$ <u>as</u> $n \to \infty$ <u>uniformly on compact subsets of</u> $-\infty < t \leq 0$. <u>If for any</u> f <u>in</u> Y <u>the solution</u> $X(t,f)$ <u>of</u> (L_∞) <u>is bounded on</u> R^+, <u>then the trivial solution of (L) is uniformly stable.</u>

<u>Proof.</u> As in the proof of Theorem 2 above it follows that $\{A_t\}$ is a one parameter family of linear maps on Y into R_n. The A_t are again norm bounded, that is if

$$\|A_t\|_Y = \sup\{\|A_t f\| : \|f\|_0 = 1,\ f \in Y\},$$

then $A^* = \sup_t \|A_t\|_Y < \infty$. Given f in $C[-\infty,0]$, let f_n be the approximating sequence guaranteed by the hypotheses. Then for any t and any $T > 0$ one has

$$\begin{aligned} &\left|\int_0^t R(t-s)\{\int_0^\infty B(s+u)[f_n(-u)-f(-u)]du\}ds\right| \\ &\quad \leq \int_0^t |R(t-s)|\{\int_0^T |B(s+u)||f_n(-u)-f(-u)|du\} \\ &\qquad + \int_T^\infty |B(s+u)|du(\|f_n\|_0+\|f\|_0)\}ds \to 0 \end{aligned}$$

as $T, n \to \infty$. Then (3.3) implies that $X(t,f_n) \to X(t,f)$ as $n \to \infty$ for each fixed $t \geq 0$. In particular

$$|X(t,f)| \leq \lim\inf|X(t,f_n)| \leq (A^*+\sup|R(t)|)\lim\inf\|f_n\|_0 < \infty$$

so that $X(t,f)$ is bounded on R^+. Q.E.D.

Theorem 2 has the following corollary.

__Corollary 1.__ __Suppose__ $B \in L^1(R^+)$. __If the trivial solution of__ (L) __is uniformly stable, then determinant of__ $s-A-B^*(s) \neq 0$ __when__ $\operatorname{Re} s > 0$.

__Proof.__ Suppose that there exists a complex number s_0 and a unit vector x_0 such that $\operatorname{Re} s_0 > 0$ and $(s_0-A-B^*(s_0))x_0 = 0$. If one defines $X(t) = \exp(s_0 t)x_0$, then $X(t)$ is bounded on $-\infty < t \leq 0$ and

$$X'(t) - AX(t) - \int_{-\infty}^t B(t-s)X(s)ds = \{s_0-A-B^*(s_0)\}x_0\exp(s_0 t) = 0$$

for all t. Since $X(t)$ becomes unbounded as $t \to \infty$, then by Theorem 2, parts i) and iii) it follows that (L) is not uniformly stable. Q.E.D.

The same type of analysis is available for studying asymptotic stability. Three results of this type are quoted. Their proofs are similiar to the proofs given above.

Theorem 4. *Let $B \in L^1(R^+)$ and suppose that (L) is uniformly stable. Then the following statements are equivalent:*

i) *the trivial solution of (L) is uniformly asymptotically stable.*

ii) *given $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that the solution $X(t,F)$ of (L_∞) with initial value F at $\tau = 0$ satisfies the bound $|X(t+T(\varepsilon),F)| \leq \varepsilon$ uniformly for all $t \geq 0$ and all $F \in C[-\infty,0]$ with $\|F\|_0 \leq 1$.*

Theorem 5. *Let Y be a closed supspace of $C[-\infty,0]$ which satisfies the approximation condition of Theorem 3. Then in Theorem 4 one can replace the condition "all $F \in C[-\infty,0]$" by "all F in Y".*

Corollary 2. *Suppose $B \in L^1(R^+)$. If the trivial solution of (L) is uniformly asymptotically stable, then the determinant $(s-A-B^*(s)) \neq 0$ whenever $\operatorname{Re} s \geq 0$.*

IV. L^1-Properties of Solutions

The purpose of this section is to prove the most important result of the paper, namely the equivalence of uniform asymptotic stability of (L) with $R(t)$ in $L^1(0,\infty)$. Half of this assertion is easy.

Theorem 6. *Suppose $B(t)$ and $R(t)$ are both in $L^1(R^+)$. Then*

i) *$R' \in L^1(R^+)$ and both $R(t)$ and $R'(t) \to 0$ as $t \to \infty$,*

ii) *the trivial solution of (L) is uniformly asymptotically stable, and*

iii) *for any initial value (τ, f) in $R^+ \times C(R^+)$ the solution $x(t,\tau,f)$ of (L) is in $L^1(\tau,\infty)$.*

Proof. Since A is a constant matrix, then $AR(t) \in L^1(R^+)$. Moreover, the convolution of two L^1 functions results in an L^1 function (by Fubini's theorem). These two facts plus the resolvent equation (R) imply that $R'(t)$ is in $L^1(R^+)$. Since R' is in $L^1(R^+)$, then $R(t)$ has a limit at $t = \infty$. But $R \in L^1(R^+)$ so this limit is zero. To see that $R'(\infty) = 0$ note that the convolution of an L^1 function with a function which tends to zero at $t = \infty$ yields a function which is zero at infinity. This may be used in (R) to see that $R'(\infty) = 0$.

To prove part ii) note that by (2.2) one has

$$x(t+\tau,\tau,f) = R(t)f(\tau) + \int_0^t R(t-s)\{\int_0^\tau B(s+u)f(\tau-u)du\}ds.$$

Therefore, one can estimate

$$\begin{aligned}|x(t+\tau,\tau,f)| &\leq |R(t)|\ |f(\tau)| + \int_0^t |R(t-s)|\{\int_0^\tau |B(s+u)|du\}ds\|f\|_\tau \\ &\leq \|f\|_\tau\{|R(t)| + \int_0^t |R(t-s)|\{\int_s^\infty |B(u)|du\}ds\}.\end{aligned}$$

The first term $|R(t)| \to 0$ as $t \to \infty$. The second term is the convolution of an L^1 function with one which tends to zero at $t = \infty$. Therefore, the expression

$$|R(t)| + \int_0^t |R(t-s)| \{\int_s^\infty |B(u)|du\} ds$$

is bounded and tends to zero as $t \to \infty$. This proves the uniform asymptotic stability of (L).

To prove the last assertion note that (2.2) implies that for any $T > 0$ one has

$$\int_0^T |x(t+\tau,\tau,f)|dt \leq |f(\tau)| \int_0^T |R(t)|dt + \int_0^T \int_0^t |R(t-s)| \int_0^\tau |B(s+u)| \; |f(\tau-u)| dudsdt$$

$$\leq |f(\tau)| \int_0^\infty |R(t)|dt + \int_0^\tau \int_0^T \int_s^T |R(t-s)| \; |B(s+u)| \; |f(\tau-u)| dtdsdu$$

$$\leq |f(\tau)| \int_0^\infty |R(t)|dt + \int_0^\tau \{\int_0^\infty |R(t)|dt \int_0^\infty |B(s)|ds\} |f(\tau-u)|du.$$

Since $T > 0$ is arbitrary, iii) is proved. Q.E.D.

The proof of the other half of the equivalence depends on the theory of Lyapunov functionals.

Definition 5. A Lyapunov functional is a continuous function $V: R^+ \times C(R^+) \to R^1$ such that $V(t,f)$ is locally Lipschitz continuous in f. The derivative of V w.r.t. a system of equations

(4.1) $$y'(t) = F(t,y), \qquad F: R^+ \times C(R^+) \to R^n$$

is defined by

$$\dot{V}(t,f) = \limsup_{h \to 0^+} \{V(t+h,f^*) - V(t,f)\}/h$$

where

$$f^*(s) = \begin{cases} f(s) & \text{on } 0 \leq s \leq t \\ f(t) + F(t,f)(s-t) & \text{on } t \leq s \leq t+h \end{cases}.$$

Lemma 2. Let $V(t,f)$ and $F(t,f)$ be continuous and locally Lipschitz continuous in f. Then for any (τ,f) in $R^+ \times C(R^+)$ the derivative of V w.r.t. (4.1) may be written in the form

$$\dot{V}(\tau,f) = \limsup_{h \to 0^+} \{V(\tau+h, y(\cdot,\tau,f)) - V(\tau,f)\}/h$$

where $y(\cdot,\tau,f)$ is the unique solution of (4.1) with initial values (τ,f). Moreover, let $P(t,f)$ be any continuous, nonanticipative perturbation and $Y(t,\tau,f)$ any solution of the problem

$$Y'(t) = F(t,Y) + P(t,Y)$$

with initial values (τ,f). Given constants A and $B > 0$ let L be the local Lipschitz constant for $V(t,f)$ on $\{0 \leq t \leq A, \|f\|_t \leq B\}$. If $0 \leq \tau \leq A$ and $\|f\|_\tau \leq B$ then

$$\limsup_{h \to 0^+} \{V(\tau+h, Y(\cdot,\tau,f)) - V(\tau,f)\}/h \leq \dot{V}(t,f) + L|P(t,f)|.$$

The proof is similar to the corresponding proof for the ordinary differential equation's case. See Driver [2] for more details.

Theorem 7. Suppose $B \in L^1(R^+)$ and suppose the trivial solution of (L) is uniformly asymptotically stable. Then there exists a Lyapunov function $V(t,f)$ with the following properties.

i) $V(t,f)$ is locally Lipschitz continuous in f uniformly in t,

ii) $V(t,0) \equiv 0$ for all $t \geq 0$,

iii) $V(t,f) \geq \omega_0(|f(t)|)$ where $\omega_0(y)$ is a positive definite continuous function, and

iv) the derivative of V w.r.t. (L) satisfies

$$\dot{V}(t,f) \leq -\omega_1(|f(t))|)$$

where $\omega_1(y)$ is a continuous positive definite function.

Proof. The proof is essentially the same as the proof of the converse theorem of Massera [3, Theorem 8]. Pick numbers K and T_m such that if $\|f\|_\tau \leq 1$, then $|x(t,\tau,f)| \leq K$ for all $t \geq \tau$ and $|x(t+T_m+\tau,\tau,f)| < 1/m$ for all $t \geq 0$. Let $g(t)$ be a continuous, nonincreasing, positive function such that $g(t) = K$ on $0 \leq t \leq T_1$ and $g(T_m) = 1/(m-1)$ for $m = 2,3,4,\ldots$. Then $|x(t+\tau,\tau,f)| \leq g(t) \to 0$ as $t \to \infty$ whenever $\tau \geq 0$ and $\|f\|_\tau \leq 1$. For this $g(t)$ there exists a function $G(y) \in C^1(R^+)$ such that $G(y) > 0$, $G'(y) > 0$ for all $y > 0$, $G(0) = G'(0) = 0$, $G'(y)$ is increasing in y and for any constant $C > 0$ the integrals

$$\int_0^\infty G(Cg(s))ds \quad \text{and} \quad \int_0^\infty G'(Cg(s))ds$$

are finite (see Massera [3, p. 716]). Define

$$V(t,f) = \int_0^\infty G(|x(s+t,t,f)|)ds$$

where $x(t,\tau,f)$ is the unique solution of (L) with initial values (τ,f).

Since $x(t,\tau,f)$ is continuous on $R^+ \times R^+ \times C(R^+)$ and the integral in the definition of V converge uniformly for $t \geq 0$ and $\|f\|_t \leq A$, for any fixed $A > 0$, then clearly $V\colon R^+ \times C(R^+) \to R^+$ is continuous and nonanticipative. In order to see that $V(t,f)$ locally Lipschitz continuous in f uniformly in $t \geq 0$ fix any constant $B > 0$. Pick f_1 and f_2 with $\|f_1\|_t$ and $\|f_2\|_t$ both less than or equal to B. Since $G'(y)$ is increasing, then for any pair of vectors a and b one has

$$\begin{aligned} |G(|a|) - G(|b|)| &\leq G'(q|a| + (1-q)|b|)(|a|-|b|) \quad (0 < q < 1) \\ &\leq G'(|a|+|b|)|a-b|. \end{aligned}$$

By stability there exists a constant $K > 0$ such that if $\|\varphi\|_t \leq 1$ then $|x(s+t,t,\varphi)| \leq K$ for all $t,s \geq 0$. Thus one has

$$\begin{aligned} |V(t,f_1) - V(t,f_2)| &\leq \int_0^\infty |G(|x(s+t,t,f_1)|) - G(|x(s+t,t,f_2|)|ds \\ &\leq \int_0^\infty G'(|x(s+t,t,f_1)| + |x(s+t,t,f_2)|)|x(s+t,t,f_1-f_2)|ds \\ &\leq \int_0^\infty G'(2Bg(s))K\|f_1-f_2\|_t ds \\ &= \{K \int^\infty G'(2Bg(S))ds\}\|f_1-f_2\|_t. \end{aligned}$$

This proves the Lipschitz continuity.

To see that V is positive definite let $t \geq 0$ be fixed and let B be a given constant. By stability if $\|f\|_t \leq B$, then $x(s+t,t,f)$ is uniformly bounded. Hence $|x'(s+t,t,f)| \leq \alpha(B)$ is uniformly bounded for $t,s \geq 0$ and $\|f\|_t \leq B$. This means that

$$|x(s+t,t,f) - f(t)| \leq s\,\alpha(B) \leq |f(t)|/2$$

if $0 \leq s \leq |f(t)|/(2\alpha(B))$. Using this in the definition of V it follows that

$$V(t,f) \geq \int_0^{\omega(|f(t)|)} G(|f(t)|/2)dt = \omega_0(|f(t)|)$$

if $\|f\|_t \leq B$ and $\omega(y) = y/\{2\alpha(B)\}$.

Finally, note that

$$\begin{aligned} V(t,x(t,\tau,f)) &= \int_0^{\infty} G(|x(s+t,t,x(t,\tau,f))|)ds \\ &= \int_0^{\infty} G(|x(s+t,\tau,f)|)ds \\ &= \int_t^{\infty} G(|x(s,\tau,f)|)ds. \end{aligned}$$

Therefore, the derivative of V w.r.t. (L) is

$$\dot{V}(t,f) = -G(|f(t)|) = -\omega_1(|f(t)|).$$

This function is negative definite. Q.E.D.

Theorem 8. *Suppose* $B \in L^1(R^+)$ *and* (L) *is uniformly asymptotically stable. Then the perturbed equation* (2.1) *has the following type of stability: Given any* $\varepsilon > 0$ *there exist two positive numbers* η_1 *and* η_2 *such that for any initial values* (τ, f), *if* $\|f\|_\tau \leq \eta_1$ *and if* $P(t,f)$ *is any continuous, nonanticipative function with* $|P(t,f)| \leq \eta_2$ *on the set* $\{t \geq 0, \|f\|_t \leq \varepsilon\}$, *then any solution* $y(t,\tau,f,P)$ *of* (2.1) *exists and satisfies* $|y(t,\tau,f,P)| < \varepsilon$ *for all* $t \geq \tau$.

Proof. Given $\varepsilon > 0$, let $\omega_0(y)$ and $\omega_1(y)$ be the positive definite functions given in Theorem 7 and let L be the Lipschitz constant for $V(t,f)$ when $\|f\|_t \leq \varepsilon$. Define $m = \min\{\omega_0(y): |y| = \varepsilon\}$. Since $V(t,f) \leq L\|f\|_t$ when $\|f\|_t \leq \varepsilon$, then $V(t,f) \leq m$ if $\|f\|_t \leq \eta_1$ and $\eta_1 = \min\{\varepsilon/2, m(2L)^{-1}\}$. Let $\alpha = \min\{\omega_1(y) = \eta_1 \leq |y| \leq \varepsilon\}$ and set $\eta_2 = \alpha/(2L)$. This choice of η_1 and η_2 will do.

Let f and P be majorized by η_1 and η_2 and let $\varphi(t) = y(t+\tau,\tau,f,P)$. Since $\varphi(t)$ is continuous and $|\varphi(0)| = |f(0)| \leq \eta_1 < \varepsilon$, then $|\varphi(t)| < \varepsilon$ for t sufficiently small. If $\varphi(t)$ gets into the region $\eta_1 \leq |\varphi(t)| \leq \varepsilon$, then in this region the derivative $\dot{V}_p$ of V w.r.t. (2.1) satisfies

$$\begin{aligned}
\dot{V}_P(t+\tau,\varphi) &\leq \dot{V}_L(t+\tau,\varphi) + L|P(t+\tau,\varphi)| \\
&\leq -\omega_1(|\varphi(t)|) + L\eta_2 \\
&\leq -\alpha + L\alpha/(2L) = -\alpha/2 < 0.
\end{aligned}$$

This means that $V(t+\tau,\varphi)$ is decreasing in this region. In particular, $V(t+\tau,\varphi) \leq \max\{V(t,f): t \geq 0, \|f\|_t \leq \eta_1\} < m$ so that $|\varphi(t)| < \varepsilon$. Since $|\varphi(t)|$ can never reach the circle $|\varphi(t)| = \varepsilon$, the proof is complete. Q.E.D.

Corollary 3. *If* $B \in L^1(R^+)$ *and if* (L) *is uniformly asymptotically stable, then the resolvent* $R(t)$ *associated with the linear system* (L) *is of class* $L^1(R^+)$.

Proof. The solution of the inhomogeneous problem

$$y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds + F(t), \quad y(0) = 0$$

is given by

$$y(t) = \int_0^t R(t-s)F(s)ds \qquad (t \geq 0). \tag{4.2}$$

If F is bounded and continuous on R^+, then by Theorem 8 (with $P(t,f) = \varepsilon F(t)$ and with ε sufficiently small) it follows that $\varepsilon y(t)$ is bounded on R^+. Thus $y(t)$ is bounded. Since $y(t)$ is given by (4.2), the conclusion of the corollary follows from a result of Corduneanu [4, Theorem 3]. Q.E.D.

V. Some Consequences of Condition (D).

Let $B \in L^1(R^+)$, let $B^*(s)$ be the Laplace transform of B and assume that

$$\det(s-A-B^*(s)) \neq 0 \quad \text{when} \quad \operatorname{Re} s \geq 0. \tag{D}$$

Condition (D) is certainly necessary for uniform asymptotic stability of (L). Here we seek additional conditions on B which insure uniform stability or uniform asymptotic stability of (L).

<u>Theorem 9</u>. <u>Suppose</u> $B \in L^1(R^+)$ <u>and suppose</u> (D) <u>is true</u>.

i) <u>If for some</u> p <u>in the interval</u> $1 \leq p \leq 2$ <u>one has</u>

$$\int_0^\infty \left(\int_s^\infty |B(u)|\,du\right)^p ds < \infty,$$

<u>then</u> (L) <u>is uniformly asymptotically stable</u>.

ii) <u>If there exists a</u> p <u>in</u> $1 < p \leq 2$ <u>such that</u>

$$\int_0^\infty \left\{\int_s^\infty |B(u)|^p du\right\}^{1/p} ds < \infty,$$

<u>then the trivial solution of</u> (L) <u>is uniformly stable and asymptotically stable</u>.

<u>Proof</u>. By Lemma 1 above the transforms $R^*(i\tau)$ and $(R')^*(i\tau)$ are of class $L^p(-\infty,\infty)$ for $1 < p \leq 2$. The Fourier transforms of these functions are $2\pi R(-t)$ and $2\pi R'(-t)$ on $-\infty < t \leq 0$ and zero on $0 \leq t < \infty$. But the Fourier transform of an L^p function with $1 < p \leq 2$ is of class $L^q(-\infty,\infty)$ where $1/p + 1/q = 1$, see Titchmarsh [5, p. 96]. Therefore, R and $R' \in L^q(0,\infty)$ for all q in the interval $2 \leq q < \infty$. This implies that $R(t) \to 0$ as $t \to \infty$. The argument used in the proof of Theorem 5 shows that $R'(t) \to 0$ as $t \to \infty$.

Let s and t be numbers bigger than one and let r be the solution of the equation

$$1/r = 1/s + 1/t - 1.$$

If $r \geq 1$, then the convolution of a function in $L^s(R^+)$ with a function in $L^t(R^+)$ yields a function in $L^r(R^+)$, see Titchmarsh[5, p. 97]. If $r = +\infty$, the resulting function is in $L^\infty(R^+)$ and tends to zero as $t \to \infty$, see e.g., Rudin [6, p. 4, part (d)].

To prove i) we use Theorem 4. Let $X(t,F)$ be a solution of (L_∞) with initial value F at $\tau = 0$, that is

$$X(t,F) = R(t)F(0) + \int_0^t R(t-s)\{\int_s^\infty B(u)F(u-s)du\}ds.$$

Then one has

$$|X(t,F)| \leq |R(t)|\ |F(0)| + \int_0^t |R(t-s)| \int_s^\infty |B(u)|duds\|F\|_0$$

$$\leq \{|R(t)| + \int_0^t |R(t-s)| \int_s^\infty |B(u)|duds\}\|F\|_0.$$

We know that $|R(t)| \to 0$ as $t \to \infty$. If

$$b(s) = \int_s^\infty |B(u)|du$$

is in $L^p(R^+)$ for some p in $1 < p \leq 2$, then since $R \in L^q(R^+)$ for $q = p/(p-1)$, then the convolution is bounded and tends to zero as $t \to \infty$. If $b(s)$ is of class $L^1(R^+)$, then since $R(t) \to 0$ as $t \to \infty$, the convolution still tends to zero as $t \to \infty$. Thus (L) is uniformly asymptotically stable.

To prove ii) first note that if $1/p + 1/q = 1$, then

$$Z(t) = \int_0^t |R(t-s)| \int_s^\infty |B(u)| du ds = \int_0^\infty \int_0^t |R(t-s)| \ |B(s+u)| ds du$$

$$\leq \int_0^\infty \{\int_0^t |R(t-s)|^q ds\}^{1/q} \{\int_0^t |B(s+u)|^p ds\}^{1/p} du$$

$$\leq \int_0^\infty \{\int_0^\infty |R(s)|^q ds\}^{1/q} \{\int_u^\infty |B(s)|^p ds\}^{1/p} du.$$

Thus $Z(t)$ is uniformly bounded on R^+. Since $Z(t) \geq y(t)$, then by Theorem 1 it follows that (L) is uniformly stable. To see that (L) is asymptotically stable consider a solution,

$$x(t+\tau,\tau,f) = R(t)f(\tau) + \int_0^t R(t-s) \int_0^\tau B(s+u)f(\tau-u) du ds.$$

We know that $R(t) \to 0$ as $t \to \infty$. Consider the second term

$$x_0(t) = \int_0^t R(t-s)\{\int_0^\tau B(s+u)f(\tau-u)du\}ds.$$

If $q = p/(p-1)$, then

$$|\int_0^t B(s+u)f(\tau-u)du| \leq \{\int_0^\tau |B(s+u)|^p du\}^{1/p} \{\int_0^\tau |f(\tau-u)|^q du\}^{1/q}$$

$$\leq \{\int_s^\infty |B(u)|^p du\}^{1/p} \{\int_0^\tau |f(u)|^q du\}^{1/q}.$$

Therefore, $x_0(t)$ is the convolution of $R \in L^q(R^+)$ with an L^1 function, that is $x_0(\cdot) \in L^q(R^+)$. Since $x_0(t)$ has a bounded derivative, it tends to zero as $t \to \infty$. Q.E.D.

As an example if $B(t) = \mathcal{O}(t^{-\alpha})$ as $t \to \infty$ for some constant $\alpha > 3/2$, then the hypotheses of Theorem 9, i) and ii) are satisfied. One can use any value of p in the range $(\alpha-1)^{-1} < p \leq 2$. If

$$\int_s^\infty |B(t)|dt = \mathcal{O}(s^{-\alpha}) \text{ as } s \to \infty$$

where $\alpha > \frac{1}{2}$, then the hypotheses of part i) are true.

Theorem 10. *Suppose* $B \in L^1(0,\infty)$ *and* (D) *is true. Suppose* $B^*(i\tau)$ *is locally Holder continuous with exponent* α

$$|B^*(i\tau+ih) - B^*(i\tau-ih)| \leq K(\tau)h^\alpha$$

where $K(\tau)(1+\tau^2)^{-1} \in L^2(-\infty,\infty)$. *If either* $\alpha > 1/2$ *or if* $0 < \alpha \leq 1/2$ *and there exists a number* $q \geq 2$ *such that*

$$(5.1) \qquad \int_0^\infty \{\int_s^\infty |B(u)|du\}^q ds < \infty \quad \text{and} \quad \alpha + 1/q > 1/2,$$

then (L) *is uniformly asymptotically stable. On the other hand if* $0 < \alpha \leq 1/2$ *and there exists a number* $q \geq 2$ *such that*

$$(5.2) \qquad \int_0^\infty \{\int_s^\infty |B(u)|^q du\}^{1/q} ds < \infty \quad \text{and} \quad \alpha + 1/q > 1/2,$$

then (L) *is both uniformly stable and asymptotically stable.*

<u>Proof</u>. Since $R^*(i\tau) = \{i\tau - A - B^*(i\tau)\}^{-1}$ and $B^*(i\tau) \to 0$ as $t \to \pm\infty$, then

$$R^*(i\tau+ih) - R^*(i\tau-ih) = \{2ih + B^*(i\tau-ih) - B^*(i\tau+ih)\}\,\mathcal{O}(\tau^{-2}).$$

The Holder continuity of B^* and the integrability of $K(\tau)$ imply that

$$\int_{-\infty}^{\infty} |R^*(i\tau+ih) - R^*(i\tau-ih)|^p d\tau = \mathcal{O}(h^{\alpha p})$$

as $h \to 0$ for any p in $(1,2]$. This means that the Fourier transform of $R^*(i\tau)$ is in $L^r(-\infty,\infty)$ for all r in the range

$$p(p+\alpha p-1)^{-1} < r < p(p-1)^{-1},$$

see Titchmarsh [5, p. 115]. The maximum value of the lower limit occurs when $p = 2$ so that

$$2(2\alpha+1)^{-1} < r < 2.$$

In particular if $1/2 < \alpha \leq 1$, then $R \in L^1(R^+)$ and (L) is uniformly asymptotically stable by Theorem 6 above.

Suppose $0 < \alpha \leq 1/2$. Then the function $y(t)$ defined in Theorem 1 above satisfies

$$y(t) \leq Z(t) = \int_0^t |R(t-s)|\{\int_s^{\infty} |B(u)|du\}ds.$$

But $Z(t)$ is the convolution of a function in $L^r(2(2\alpha+1)^{-1} < r < 2)$

with a function in L^q. This range of values for r includes a value for which $1/q + 1/r = 1$. Therefore, $Z(t) \to 0$ as $t \to \infty$.

On the other hand if (5.2) is true, then one can estimate

$$Z(t) = \int_0^\infty \{\int_0^t |R(t-s)|\ |B(s+u)|ds\}ds$$

$$\leq \int_0^\infty \{\int_0^t |R(t-s)|^r ds\}^{1/r}\{\int_0^t |B(s+u)|^q ds\}^{1/q} du$$

$$\leq \{\int_0^\infty |R(s)|^r ds\}^{1/r} \int_0^\infty \{\int_u^\infty |B(s)|^q ds\}^{1/q} du.$$

Since $Z(t)$ is bounded on R^+, then Theorem 1 asserts the uniform stability of (L). As in the proof of Theorem 9 one can show that $x_0(t) = x(t+\tau,\tau,f) - R(t)f(\tau)$ is in $L^q(R^+)$ and so $x_0(t) \to 0$ as $t \to \infty$. This will prove the asymptotic stability of (L). Q.E.D.

Note that in Theorem 10 the assumption that B^* is Holder continuous and $K(\tau)(1+\tau^2)^{-1} \epsilon L^2(-\infty,\infty)$ could be replaced by any other assumption which will insure that

$$\int_{-\infty}^{\infty} |R^*(i\tau+ih) - R^*(i\tau-ih)|^2 d\tau = \mathcal{O}(h^{2\alpha})$$

as $h \to 0$.

It is easy to find examples where the hypotheses of Theorem 10 may be verified. Suppose $tB(t) \epsilon L^p(0,\infty)$ for some p in $1 < p \leq 2$. Let $1/p + 1/q = 1$. Then the function $d/ds[B^*(s)] = \int_0^\infty \exp(-st)(-tB(t))dt$ is uniformly integrable L^q on each vertical line segment in $\text{Res} \geq 0$, say

$$\sup \{\int_0^\infty \left| \frac{dB^*}{ds} (\sigma+i\tau)\right|^q d\tau:\ \sigma \geq 0\} = A^q < \infty$$

(see Titchmarsh [5, p. 97, line 4.1.2]). Thus for any $\sigma > 0$ the Holder inequality implies that

$$(5.3) \qquad |B^*(\sigma+i\tau+ih) - B^*(\sigma+i\tau-ih)| \leq A(2h)^\alpha$$

on $-\infty < \tau < \infty$ where h is any positive constant and $\alpha = 1/p$. Since

$$\int^\infty |B(t)|dt = \int^\infty |tB(t)|\cdot t^{-1}dt \leq \{\int^\infty |tB(t)|^p dt\}^{1/p}\{\int^\infty t^{-q}dt\}^{1/q} < \infty,$$

then $B \in L^1(0,\infty)$. Thus $B^*(s)$ is continuous on the half plane $\operatorname{Re} s \geq 0$. The continuity of $B^*(s)$ allows one to take the limit as $\sigma \to 0^+$ in (5.3). It follows that (5.3) remains true when $\sigma = 0$.

For example, if $tB(t) \in L^2(R^+)$, then (5.3) is true with $\alpha = 1/2$. In addition, one has

$$\begin{aligned}
\int^\infty (\int_s^\infty |B(u)|du)^2 ds &= \int^\infty (\int_s^\infty |uB(u)|\cdot u^{-1}du)^2 ds \\
&\leq \int^\infty (\int_s^\infty |uB(u)|^2 du)(s^{-3}/3)ds \\
&\leq (\int_0^\infty |uB(u)|^2 du) \int^\infty (s^{-3}/3)ds < \infty.
\end{aligned}$$

Therefore, (5.1) is true with $\alpha = 1/2$ and $q = 2$.

VI. Asymptotic Stability.

The purpose of this section is to prove the following theorem.

Theorem 11. *Let* $B \in L^1(R^+)$ *and let* (D) *be true. Then the trivial solution of* (L) *is asymptotically stable. Moreover, for any initial value* (τ, f) *in* $R^+ \times C(R^+)$ *the solution* $x(t,\tau,f)$ *of* (L) *is of class* $L^q[\tau,\infty)$ *for each* $q \in [2,\infty)$.

Proof. Let $x(t) = x(t+\tau,\tau,f)$ so that

$$(6.1) \qquad x(t) = R(t)f(\tau) + \int_0^t R(t-s)\{\int_0^\tau B(s+u)f(\tau-u)du\}ds,$$

for all $t \geq 0$. From Lemma 1 we know that R and $R' \in L^q(R^+)$ for all $q \geq 2$, and that $R(t) \to 0$ as $t \to \infty$. The Laplace transform of the second term in (6.1) is

$$\varphi(s) = R^*(s) \int_0^\tau \{\int_0^\infty \exp(-st)B(t+u)dt\}f(\tau-u)du,$$

where $R^*(s)$ is the transform of the resolvent $R(t)$. By Lemma 1 above one has

$$|\varphi(s)| \leq K(1+|s|)^{-1} \int_0^\tau \{\int_0^\infty |B(t)|dt\}|f(\tau-u)|du$$

$$\leq K(1+|s|)^{-1} \int_0^\infty |B(t)|dt \int_0^\tau |f(u)|du.$$

In particular, then $|\varphi(i\tau)| \in L^p(-\infty,\infty)$ for any p in $(1,2]$. The Fourier transform of $\varphi(i\tau)$ is then of class $L^q(-\infty,\infty)$ for all q in $2 \leq q < \infty$

(c.f. [5, p. 97]). This Fourier transform is essentially the second term in (6.1). Therefore, $x(t) = x(t+\tau,\tau,f) \in L^q(0,\infty)$ for $2 \leq q < \infty$.

When $p = q = 2$ in the analysis above, then the term

$$Z(t) = \int_0^t (t-s)\{\int_0^\tau B(s+u)f(\tau-u)du\}ds$$

is in $L^2(0,\infty)$ and

$$\int_0^\infty |Z(t)|^2 dt \leq \sqrt{2\pi}\, K \int_0^\infty |B(t)|\, dt \int_0^\tau |f(u)|\, du \int_{-\infty}^\infty (1+\tau^2)^{-1} d\tau$$

$$= K_1 \int_0^\tau |f(u)|\, du,$$

where K_1 is a fixed constant independent of τ and f. By (6.1) it follows that $|x(t)| \leq |R(t)|\,|f(\tau)| + |Z(t)| \in L^2(0,\infty)$. Since

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + \int_0^\tau B(s+\tau)f(\tau-u)du,$$

then $x'(t)$ is the sum of two L^2 functions and a function which tends to zero as $t \to \infty$. Thus $x(t)$ is also uniformly continuous on R^+ and $x(t) \to 0$ as $t \to \infty$.

In order to prove stability note that

$$|x(t)| \leq |R(t)|\ |f(\tau)| + |\int_0^\tau \{\int_0^t R(t-s)B(s+u)ds\} f(\tau-u)du|$$

$$\leq |R(t)|\ |f(\tau)| + \sup_{t \geq 0} |R(t)| \int_0^\tau \{\int_u^\infty |B(u)|\,du\} |f(\tau-u)|\,du$$

$$\leq \sup_{t \geq 0} \; |R(t)| \{|f(\tau)| + \int_0^\infty |B(u)| du \int_0^\tau |f(u)| du\}.$$

This shows that $x(t)$ is uniformly small when $f(u)$ is small uniformly on $0 \leq u \leq \tau$. Q.E.D.

The asymptotic stability could also be proved by showing that the functional

$$V(t,f) = \int_0^\infty |x(s+t,t,f)|^q ds, \quad (t,f) \in R^+ \times C(R^+)$$

is a Lyapunov functional (for any q in $[2,\infty)$) which satisfies the hypotheses of an asymptotic stability theorem of Driver [2, Theorem 6].

VII. Some Examples and Questions.

In order to obtain examples which satisfy (D) one can pick any function $B \in L^1(R^+)$. Then pick a constant $A < -\int_0^\infty |B(t)| dt$. It follows that (D) is always true. To find an example with $A = 0$, let

$$B(t) = -\sum_{n=0}^{\infty} a_n \exp(-b_n t)$$

where the a_n and b_n are positive and where

$$\sum_{n=0}^{\infty} a_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} (a_n/b_n) < \infty.$$

The convergence of these series is sufficient to insure that B is in $L^1(R^+)$. Then

$$-B^*(s) = \sum_{n=0}^{\infty} a_n (s+b_n)^{-1},$$

so that if $s = \sigma + i\tau$ and $\sigma \geq 0$ one has

$$\mathrm{Re}(s-B^*(s)) = \sigma + \sum_{n=0}^{\infty} a_n(\sigma+b_n)|s+b_n|^{-2} > 0.$$

Therefore, $s - B^*(s) \neq 0$ when $\mathrm{Re}\, s \geq 0$. Since the a_n and b_n are all positive it is easy to compute certain integrals involving $B(t)$. For example

$$\int_0^{\infty} |tB(t)|^2 dt = 2 \sum_{n,m=0}^{\infty} a_n a_m (b_n+b_m)^{-3}.$$

If this last sum is finite, then Theorem 10 applies.

The results in this paper suggest several interesting questions. For example, in Theorem 9, part i) is the conclusion still true if p is in the range $2 < p < \infty$? Similarly in part ii) is the conclusion true if $2 < p < \infty$? In Theorem 9, part ii) can the conclusion be strengthened to uniform asymptotic stability? If these results are true, then are the hypotheses of Theorem 11 sufficient for uniform asymptotic stability?

Hannsgen [7] has given sufficient conditions on the coefficients A and $B(t)$ in (L) so that the resolvent satisfies $R^*(i\tau) \in L^p(-\infty,\infty)$ for $1 < p \leq 2$ even though $B(t)$ is not in $L^1(R^+)$. Under Hannsgen's assumptions it would be interesting to see what types of stability are present. Even more important, can one show that $R(t)$ or $R'(t) \in L^1(R^+)$?

Bibliography

1. S.I. Grossman and R.K. Miller, Perturbation theory for Volterra Integro-differential systems, J. Diff. Equations 8(1970), 457-474.

2. R. D. Driver, Existence and stability of solutions of a delay-differential system, Arch. Rat. Mech. Anal. 10(1962), 401-426.

3. J. L. Massera, On Liapunov's conditions of stability, Ann. Math. 50(1949), 705-721.

4. C. Corduneanu, Problèmes globaux dans la théorie des équations intégrales de Volterra, Ann. Mat. Pura. Appl. 67(1965), 349-363.

5. E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937.

6. Walter Rudin, Fourier Analysis on Groups , Interscience, N. Y., 1962

7. K. B. Hannsgen, Indirect Abelian theorems and a linear Volterra equation, Trans. A.M.S. 142(1969), 539-555.